

THE GENERIC ISOMETRY AND MEASURE PRESERVING HOMEOMORPHISM ARE CONJUGATE TO THEIR POWERS

CHRISTIAN ROSENDAL

ABSTRACT. It is known that there is a comeagre set of mutually conjugate measure preserving homeomorphisms of Cantor space equipped with the coin-flipping probability measure, i.e., Haar measure.

We show that the generic measure preserving homeomorphism is moreover conjugate to all of its powers. It follows that the generic measure preserving homeomorphism extends to an action of $(\mathbb{Q}, +)$ by measure preserving homeomorphisms.

Similarly, S. Solecki has proved that there is a comeagre set of mutually conjugate isometries of the rational Urysohn metric space. We prove that these are all conjugate with their powers and hence therefore also embed into \mathbb{Q} -actions. By consequence, the generic isometry of the full Urysohn metric space has roots of all orders.

We also consider a notion of topological similarity in Polish groups and use this to give simplified proofs of the meagreness of conjugacy classes in the automorphism group of the standard probability space and in the isometry group of the Urysohn metric space.

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1. INTRODUCTION

Suppose M is a compact metric space and let $\text{Homeo}(M)$ be its group of homeomorphisms. We equip $\text{Homeo}(M)$ with the topology of uniform convergence or

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what is equivalent, since M is compact metric, the compact-open topology. Thus, in this way, a neighbourhood basis at the identity 1 consists of the sets

$$\{h \in \text{Homeo}(M) \mid h(C_1) \subseteq V_1 \ \& \ \dots \ \& \ h(C_n) \subseteq V_n\},$$

where $V_i \subseteq M$ is open and $C_i \subseteq V_i$ compact. Under this topology the group operations are continuous and thus $\text{Homeo}(M)$ is a topological group. Moreover, the topology is *Polish*, that is, $\text{Homeo}(M)$ is separable and its topology can be induced by a complete metric.

Now consider the case when M is Cantor space $2^{\mathbb{N}}$. Then, as any two disjoint closed sets in $2^{\mathbb{N}}$ can be separated by a clopen set, we get a neighbourhood basis at the identity consisting of sets of the form

$$\{h \in \text{Homeo}(2^{\mathbb{N}}) \mid h(C_1) = C_1 \ \& \ \dots \ \& \ h(C_n) = C_n\},$$

where $C_1, \dots, C_n \subseteq 2^{\mathbb{N}}$ is a partition of $2^{\mathbb{N}}$ into clopen sets.

By Stone duality, the homeomorphisms of Cantor space are just the automorphisms of the boolean algebra of clopen subsets of $2^{\mathbb{N}}$, which we denote by \mathbf{B}_{∞} . Thus, viewed in this way, the neighbourhood basis at the identity has the form

$$\{h \in \text{Homeo}(2^{\mathbb{N}}) \mid h|_{\mathbf{C}} = \text{id}_{\mathbf{C}}\}$$

where \mathbf{C} is a finite subalgebra of \mathbf{B}_{∞} .

Cantor space $2^{\mathbb{N}}$ is of course naturally homeomorphic to the Cantor group $(\mathbb{Z}_2)^{\mathbb{N}}$ and therefore comes equipped with Haar measure μ . Up to a homeomorphism of Cantor space μ is the unique atomless Borel probability measure on $2^{\mathbb{N}}$ such that

- if $C \in \mathbf{B}_{\infty}$, then $\mu(C)$ is a *dyadic rational*, i.e., on the form $\frac{n}{2^k}$,
- if $C \in \mathbf{B}_{\infty}$ and $\mu(C) = \frac{n}{2^k}$, then for all $l \geq k$, there is some clopen $B \subseteq C$ such that $\mu(B) = \frac{1}{2^l}$,
- if $\emptyset \neq C \in \mathbf{B}_{\infty}$, then $\mu(C) > 0$.

The measure μ is of course the product probability measure of the coinflipping measure on each factor $2 = \{0, 1\}$. For simplicity, we call μ Haar measure on $2^{\mathbb{N}}$.

One easily sees that the group of Haar measure preserving homeomorphisms $\text{Homeo}(2^{\mathbb{N}}, \mu)$ of $2^{\mathbb{N}}$ is a closed subgroup of $\text{Homeo}(2^{\mathbb{N}})$ and therefore a Polish group in its own right. It was proved by Kechris and Rosendal in [8] that there are comeagre conjugacy classes in both $\text{Homeo}(2^{\mathbb{N}})$ and $\text{Homeo}(2^{\mathbb{N}}, \mu)$. In fact, the result for $\text{Homeo}(2^{\mathbb{N}}, \mu)$ is rather simple and also holds for many other sufficiently homogeneous measures on $2^{\mathbb{N}}$ (see Akin [1]). This result allows us to refer to the *generic* measure preserving homeomorphism of Cantor space (with Haar measure) knowing that they are all mutually conjugate. One of the aims of this paper is to show that they are all conjugate to their non-zero powers, which will in turn show that they all are part of an action of the additive group \mathbb{Q} by measure preserving homeomorphisms of $2^{\mathbb{N}}$. Notice that this is to some extent an optimal result, for as $\text{Homeo}(2^{\mathbb{N}}, \mu)$ is totally disconnected there are no non-trivial continuous homomorphism (or even measurable homomorphisms) from \mathbb{R} into $\text{Homeo}(2^{\mathbb{N}}, \mu)$ and thus \mathbb{R} cannot act non-trivially by measure preserving homeomorphisms on $2^{\mathbb{N}}$.

The *Urysohn metric space* \mathbb{U} is a universal separable metric space first constructed by Urysohn in the posthumously published [13]. It soon went out of fashion following the discovery that many separable Banach spaces are already universal separable metric spaces, but has come to the forefront over the last twenty years as an analogue of Fraïssé theory in the case of metric spaces.

The Urysohn space \mathbb{U} is characterised up to isometry by being separable and complete together with the following extension property.

If $\phi: \mathbf{A} \rightarrow \mathbb{U}$ is an isometric embedding of a finite metric space \mathbf{A} into \mathbb{U} and $\mathbf{B} = \mathbf{A} \cup \{y\}$ is a one point metric extension of \mathbf{A} , then ϕ extends to an isometric embedding of \mathbf{B} into \mathbb{U} .

There is also a rational variant of \mathbb{U} called the *rational Urysohn metric space*, which we denote by \mathbb{QU} . This is up to isometry the unique countable metric space with only rational distances such that the following variant of the above extension property holds.

If $\phi: \mathbf{A} \rightarrow \mathbb{QU}$ is an isometric embedding of a finite metric space \mathbf{A} into \mathbb{QU} and $\mathbf{B} = \mathbf{A} \cup \{y\}$ is a one point metric extension of \mathbf{A} whose metric only takes rational distances, then ϕ extends to an isometric embedding of \mathbf{B} into \mathbb{QU} .

We denote by $\text{Iso}(\mathbb{QU})$ and $\text{Iso}(\mathbb{U})$ the isometry groups of \mathbb{QU} and \mathbb{U} respectively. These are Polish groups when equipped with the topology of pointwise convergence on \mathbb{QU} seen as a discrete set and \mathbb{U} seen as a metric space respectively. Thus, the basic neighbourhoods of the identity in $\text{Iso}(\mathbb{QU})$ are of the form

$$\{h \in \text{Iso}(\mathbb{QU}) \mid h|_{\mathbf{A}} = \text{id}_{\mathbf{A}}\},$$

where \mathbf{A} is a finite subset of \mathbb{QU} . On the other hand, the basic open neighbourhoods of the identity in $\text{Iso}(\mathbb{U})$ are of the form

$$\{h \in \text{Iso}(\mathbb{U}) \mid \forall x \in \mathbf{A} \, d(hx, x) < \epsilon\},$$

where \mathbf{A} is a finite subset of \mathbb{U} and $\epsilon > 0$.

In [12] S. Solecki proved, building on work of Herwig and Lascar [6], the following result.

Theorem 1. *Let \mathbf{A} be a finite rational metric space. Then there is a finite rational metric space \mathbf{B} containing \mathbf{A} and such that any partial isometry of \mathbf{A} extends to a full isometry of \mathbf{B} .*

This in turn has the consequence that $\text{Iso}(\mathbb{QU})$ has a comeagre conjugacy class and we can therefore refer to its elements as *generic* isometries of \mathbb{QU} . The second aim of our paper is to prove that these are all conjugate to their non-zero powers, which again suffices to show that they all are part of an action of the additive group \mathbb{Q} by isometries of \mathbb{QU} .

In the last section we briefly consider a coarse notion of conjugacy in Polish groups. We say that f and g belonging to a Polish group G are *topologically similar* if for all increasing sequences (s_n) we have $f^{s_n} \xrightarrow[n \rightarrow \infty]{} 1$ if and only if $g^{s_n} \xrightarrow[n \rightarrow \infty]{} 1$. As opposed to automorphism groups of countable structures there tend not to be comeagre conjugacy classes in large connected Polish groups and we shall provide new simple proofs of this for $\text{Aut}([0, 1], \lambda)$ and $\text{Iso}(\mathbb{U})$ by showing that in fact their topological similarity classes are meagre.

2. POWERS OF GENERIC MEASURE PRESERVING HOMEOMORPHISMS

2.1. Free amalgams of measured boolean algebras. Suppose $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_n$ are finite boolean algebras containing a common subalgebra \mathbf{A} . We define the *free*

amalgam

$$\otimes_{\mathbf{A}}^l \mathbf{B}_l = \mathbf{B}_1 \otimes_{\mathbf{A}} \mathbf{B}_2 \otimes_{\mathbf{A}} \dots \otimes_{\mathbf{A}} \mathbf{B}_n$$

of $\mathbf{B}_1, \dots, \mathbf{B}_n$ over \mathbf{A} as follows.

By renaming, we can suppose that $\mathbf{B}_i \cap \mathbf{B}_j = \mathbf{A}$ for all $i \neq j$. We then take as our atoms the set of formal products

$$b_1 \otimes \dots \otimes b_n,$$

where each b_i is an atom in \mathbf{B}_i and such that for some atom a of \mathbf{A} we have $b_i \leq a$ for all i . Also, for simplicity, if $c_i \in \mathbf{B}_i$ is not necessarily an atom, but nevertheless we have some atom a of \mathbf{A} such that $c_i \leq a$ for all i , we write

$$c_1 \otimes \dots \otimes c_n = \bigvee \{b_1 \otimes \dots \otimes b_n \mid b_i \text{ is an atom in } \mathbf{B}_i \text{ and } b_i \leq c_i\}.$$

We can now embed each \mathbf{B}_i into $\otimes_{\mathbf{A}}^l \mathbf{B}_l$ by defining for each $b \in \mathbf{B}_i$ minorising an atom $a \in \mathbf{A}$

$$\pi_i(b) = a \otimes \dots \otimes a \otimes b \otimes a \otimes \dots \otimes a,$$

where the b appears in the i 'th position. In particular,

$$\pi_i(a) = a \otimes \dots \otimes a$$

for all atoms a of \mathbf{A} . Thus, for each i , $\pi_i: \mathbf{B}_i \hookrightarrow \otimes_{\mathbf{A}}^l \mathbf{B}_l$ is an embedding of boolean algebras and if $\iota_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$ denotes the inclusion mapping, then the following diagram commutes

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\iota_i} & \mathbf{B}_i \\ \iota_j \downarrow & & \downarrow \pi_i \\ \mathbf{B}_j & \xrightarrow{\pi_j} & \otimes_{\mathbf{A}}^l \mathbf{B}_l \end{array}$$

Now, if μ_i are measures on \mathbf{B}_i agreeing on \mathbf{A} , then we can define a new measure μ on $\otimes_{\mathbf{A}}^l \mathbf{B}_l$ by setting for all $b_i \in \mathbf{B}_i$ minorising the same atom $a \in \mathbf{A}$

$$\mu(b_1 \otimes \dots \otimes b_n) = \frac{\mu_1(b_1) \cdots \mu_n(b_n)}{\mu_1(a)^{n-1}}.$$

Thus,

$$\begin{aligned} \mu(\pi_i(b)) &= \mu(a \otimes \dots \otimes a \otimes b \otimes a \otimes \dots \otimes a) \\ &= \frac{\mu_1(a) \cdots \mu_{i-1}(a) \mu_i(b) \mu_{i+1}(a) \cdots \mu_n(a)}{\mu_1(a)^{n-1}} \\ &= \frac{\mu_1(a) \cdots \mu_1(a) \mu_i(b) \mu_1(a) \cdots \mu_1(a)}{\mu_1(a)^{n-1}} \\ &= \mu_i(b). \end{aligned}$$

So $\pi_i: (\mathbf{B}_i, \mu_i) \rightarrow (\otimes_{\mathbf{A}}^l \mathbf{B}_l, \mu)$ is an embedding of measured boolean algebras.

A special case is when \mathbf{A} and each \mathbf{B}_i are *equidistributed dyadic* algebras, i.e., have 2^k atoms each of measure 2^{-k} for some $k \geq 0$. Then this implies that for each i , all atoms of \mathbf{A} are the join of the same number of atoms of \mathbf{B}_i , namely, 2^{k_i-m} , where \mathbf{A} has 2^m atoms and \mathbf{B}_i has 2^{k_i} atoms. In this case, one can verify that $\otimes_{\mathbf{A}}^l \mathbf{B}_l$ has $2^{k_1+\dots+k_n-(n-1)m}$ atoms each of measure $2^{(n-1)m-k_1-\dots-k_n}$. So again this is an equidistributed dyadic algebra.

A similar construction works for *equidistributed* algebras, i.e., those having a finite number of atoms of the same (necessarily rational) measure. In this case, the amalgam is also equidistributed.

In general, an automorphism of a finite boolean algebra arises from a permutation of the atoms, but in the case of equidistributed (dyadic) algebras, any permutation of the atoms conversely gives rise to a measure preserving automorphism. Thus, for equidistributed algebras an automorphism is necessarily a measure preserving automorphism and we can therefore be a bit forgetful about the measure.

Lemma 2. *Let \mathbf{A} be an equidistributed (dyadic) finite boolean algebra. Then any partial measure preserving automorphism of \mathbf{A} extends to an automorphism of \mathbf{A} .*

Proof. Suppose that \mathbf{B} and \mathbf{C} are subalgebras of \mathbf{A} and $g: \mathbf{B} \rightarrow \mathbf{C}$ a measure preserving isomorphism. Then if b is an atom of \mathbf{B} , we have, as g is measure preserving, that b and $g(b)$ are composed of the same number of atoms of \mathbf{A} . Therefore, we can extend g to an automorphism of \mathbf{A} by choosing a bijection between the constituents of b and $g(b)$ for each atom b of \mathbf{B} . \square

2.2. Roots of measure preserving homeomorphisms.

Proposition 3. *Suppose $\mathbf{A} \subseteq \mathbf{B}$ are equidistributed (dyadic) boolean algebras, g an automorphism of \mathbf{A} and f an automorphism of \mathbf{B} such that $f|_{\mathbf{A}} = g^n$. Then there is an equidistributed (dyadic) algebra $\mathbf{C} \supseteq \mathbf{B}$ and an automorphism h of \mathbf{C} extending g and such that $h^n|_{\mathbf{B}} = f$.*

Proof. Enumerate the atoms of \mathbf{A} as a_1, \dots, a_m and the atoms of \mathbf{B} as

$$b_1^1, b_1^2, \dots, b_1^k, b_2^1, b_2^2, \dots, b_2^k, \dots, b_m^1, b_m^2, \dots, b_m^k,$$

where

$$a_i = b_i^1 \vee b_i^2 \vee \dots \vee b_i^k.$$

Since g is an automorphism of \mathbf{A} we can find a permutation ϕ of $\{1, \dots, m\}$ such that

$$g(a_i) = a_{\phi(i)}$$

for all i . Similarly, we can find a function $\psi: \{1, \dots, m\} \times \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ such that for all i and j

$$f(b_i^j) = b_{\phi^n(i)}^{\psi(i,j)}.$$

For $f(a_i) = g^n(a_i) = a_{\phi^n(i)}$ and thus $f(b_i^j) \leq f(a_i) = a_{\phi^n(i)}$, whence $f(b_i^j) = b_{\phi^n(i)}^{\psi(i,j)}$ for some $\psi(i, j) \in \{1, \dots, k\}$. Also, since

$$\begin{aligned} b_{\phi^n(i)}^{\psi(i,1)} \vee b_{\phi^n(i)}^{\psi(i,2)} \vee \dots \vee b_{\phi^n(i)}^{\psi(i,k)} &= f(b_i^1 \vee b_i^2 \vee \dots \vee b_i^k) \\ &= f(a_i) \\ &= a_{\phi^n(i)} \\ &= b_{\phi^n(i)}^1 \vee b_{\phi^n(i)}^2 \vee \dots \vee b_{\phi^n(i)}^k, \end{aligned}$$

we see that $\psi(i, \cdot): \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ is a bijection for each i .

Let $\mathbf{B}_1 = \mathbf{B}_2 = \dots = \mathbf{B}_n = \mathbf{B}$ and consider the free amalgam $\otimes_{\mathbf{A}}^l \mathbf{B}_l$. We can now define the automorphism h of $\otimes_{\mathbf{A}}^l \mathbf{B}_l$ as follows.

$$h(b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}) = b_{\phi(i)}^{\psi(i,j_n)} \otimes b_{\phi(i)}^{j_1} \otimes \dots \otimes b_{\phi(i)}^{j_{n-1}}.$$

It follows from the fact that $\psi(i, \cdot)$ is a bijection that h also is a bijection of the atoms of $\otimes_{\mathbf{A}}^l \mathbf{B}_l$ and thus defines an automorphism of $\otimes_{\mathbf{A}}^l \mathbf{B}_l$. Consider now

$$\begin{aligned} h^n(b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}) &= h^{n-1}(b_{\phi(i)}^{\psi(i, j_n)} \otimes b_{\phi(i)}^{j_1} \otimes \dots \otimes b_{\phi(i)}^{j_{n-1}}) \\ &= h^{n-2}(b_{\phi^2(i)}^{\psi(\phi(i), j_{n-1})} \otimes b_{\phi^2(i)}^{\psi(i, j_n)} \otimes b_{\phi^2(i)}^{j_1} \otimes \dots \otimes b_{\phi^2(i)}^{j_{n-2}}) \\ &= \dots \\ &= b_{\phi^n(i)}^{\psi(\phi^{n-1}(i), j_1)} \otimes b_{\phi^n(i)}^{\psi(\phi^{n-2}(i), j_2)} \otimes \dots \otimes b_{\phi^n(i)}^{\psi(i, j_n)}. \end{aligned}$$

Thus,

$$\begin{aligned} h^n(a_i \otimes a_i \otimes \dots \otimes a_i \otimes b_i^{j_n}) &= h^n\left(\bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_{n-1}=1}^k b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}\right) \\ &= \bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_{n-1}=1}^k h^n(b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}) \\ &= \bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_{n-1}=1}^k b_{\phi^n(i)}^{\psi(\phi^{n-1}(i), j_1)} \otimes b_{\phi^n(i)}^{\psi(\phi^{n-2}(i), j_2)} \otimes \dots \otimes b_{\phi^n(i)}^{\psi(i, j_n)} \\ &= a_{\phi^n(i)} \otimes a_{\phi^n(i)} \otimes \dots \otimes a_{\phi^n(i)} \otimes b_{\phi^n(i)}^{\psi(i, j_n)}. \end{aligned}$$

Similarly,

$$\begin{aligned} h(a_i \otimes a_i \otimes \dots \otimes a_i) &= h\left(\bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_n=1}^k b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}\right) \\ &= \bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_n=1}^k h(b_i^{j_1} \otimes b_i^{j_2} \otimes \dots \otimes b_i^{j_n}) \\ &= \bigvee_{j_1=1}^k \bigvee_{j_2=1}^k \dots \bigvee_{j_n=1}^k b_{\phi(i)}^{\psi(i, j_n)} \otimes b_{\phi(i)}^{j_1} \otimes \dots \otimes b_{\phi(i)}^{j_{n-1}} \\ &= a_{\phi(i)} \otimes a_{\phi(i)} \otimes \dots \otimes a_{\phi(i)}. \end{aligned}$$

We now identify \mathbf{B} with the image of \mathbf{B}_n by the embedding π_n of \mathbf{B}_n into $\otimes_{\mathbf{A}}^l \mathbf{B}_l$. Thus, the atoms of \mathbf{B} are of the form

$$a_i \otimes a_i \otimes \dots \otimes a_i \otimes b_i^j$$

and the atoms of \mathbf{A} are

$$a_i \otimes a_i \otimes \dots \otimes a_i.$$

Moreover, g acts by

$$\begin{aligned} g(a_i \otimes a_i \otimes \dots \otimes a_i) &= g(a_i) \otimes g(a_i) \otimes \dots \otimes g(a_i) \\ &= a_{\phi(i)} \otimes a_{\phi(i)} \otimes \dots \otimes a_{\phi(i)}, \end{aligned}$$

while f acts by

$$f(a_i \otimes a_i \otimes \dots \otimes a_i \otimes b_i^j) = a_{\phi^n(i)} \otimes a_{\phi^n(i)} \otimes \dots \otimes a_{\phi^n(i)} \otimes b_{\phi^n(i)}^{\psi(i, j)}.$$

Therefore, h extends g , while h^n extends f , which was what we wanted. \square

Proposition 4. *Let $n \geq 1$. Then the generic measure preserving homeomorphism of Cantor space is conjugate with its n 'th power.*

Proof. Notice first that a basic open set in $\text{Homeo}(2^{\mathbb{N}}, \mu)$ is of the form

$$U(h, \mathbf{A}) = \{g \in \text{Homeo}(2^{\mathbb{N}}, \mu) \mid g|_{\mathbf{A}} = h|_{\mathbf{A}}\},$$

where \mathbf{A} is a finite equidistributed subalgebra of \mathbf{B}_{∞} and $h \in \text{Homeo}(2^{\mathbb{N}}, \mu)$. We claim that for any $U(h, \mathbf{A})$ there is some finite equidistributed $\mathbf{B} \subseteq \mathbf{B}_{\infty}$ containing \mathbf{A} and some measure preserving homeomorphism k leaving \mathbf{B} invariant, such that $U(k, \mathbf{B}) \subseteq U(h, \mathbf{A})$. To see this, suppose h and \mathbf{A} are given. Choose an equidistributed \mathbf{B} containing both \mathbf{A} and $h(\mathbf{A})$ and notice that the partial automorphism $h: \mathbf{A} \rightarrow h(\mathbf{A})$ of \mathbf{B} extends to an automorphism \hat{h} of \mathbf{B} . So let k be any measure preserving homeomorphism of $2^{\mathbb{N}}$ that extends \hat{h} . Then \mathbf{B} is k -invariant while $U(k, \mathbf{B}) \subseteq U(h, \mathbf{A})$.

For simplicity, if k is an automorphism of a finite equidistributed algebra \mathbf{B} , we also write $U(k, \mathbf{B})$ to denote the set $\{g \in \text{Homeo}(2^{\mathbb{N}}, \mu) \mid g|_{\mathbf{B}} = k\}$.

Let now C be the comeagre conjugacy class of $\text{Homeo}(2^{\mathbb{N}}, \mu)$ and find dense open sets $V_i \subseteq \text{Homeo}(2^{\mathbb{N}}, \mu)$ such that $C = \bigcap_i V_i$. Enumerate the clopen subsets of $2^{\mathbb{N}}$ as a_0, a_1, a_2, \dots . We shall define a sequence of finite equidistributed algebras $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \dots$ of clopen sets and automorphisms g_i and f_i of \mathbf{A}_i such that

- (1) $a_i \in \mathbf{A}_{i+1}$,
- (2) g_{i+1} extends g_i ,
- (3) f_{i+1} extends f_i ,
- (4) $g_i^n = f_i$,
- (5) $U(g_{i+1}, \mathbf{A}_{i+1}) \subseteq V_i$,
- (6) $U(f_{i+1}, \mathbf{A}_{i+1}) \subseteq V_i$.

To begin, let \mathbf{A}_0 be the trivial algebra with automorphism $g_0 = f_0$. So suppose \mathbf{A}_i , g_i , and f_i are defined. We let \mathbf{B} be an equidistributed algebra containing both a_i and \mathbf{A}_i and let h be any automorphism of \mathbf{B} extending g_i . As V_i is dense open we can find some $U(k, \mathbf{C}) \subseteq V_i$, where \mathbf{C} is a k -invariant equidistributed algebra containing \mathbf{B} and k extends h . Again, as V_i is dense open, we can find some $U(p, \mathbf{D}) \subseteq V_i$, where \mathbf{D} is a equidistributed algebra containing \mathbf{C} , p a measure preserving homeomorphism leaving \mathbf{D} invariant and extending $k^n|_{\mathbf{C}}$.

Now, by Proposition 3, we can find an equidistributed algebra \mathbf{E} containing \mathbf{D} and an automorphism q of \mathbf{E} extending $k|_{\mathbf{C}}$ such that q^n extends $p|_{\mathbf{D}}$. Finally, set $\mathbf{A}_{i+1} = \mathbf{E}$,

$$g_{i+1} = q \supseteq k|_{\mathbf{C}} \supseteq h \supseteq g_i,$$

and

$$f_{i+1} = q^n \supseteq p|_{\mathbf{D}} \supseteq k^n|_{\mathbf{C}} \supseteq g_i^n = f_i.$$

Then $U(g_{i+1}, \mathbf{A}_{i+1}) \subseteq U(k, \mathbf{C}) \subseteq V_i$ and $U(f_{i+1}, \mathbf{A}_{i+1}) \subseteq U(p, \mathbf{D}) \subseteq V_i$.

Set now $g = \bigcup_i g_i$ and $f = \bigcup_i f_i$. By (1), (2), and (3), f and g are measure preserving automorphisms of \mathbf{B}_{∞} and, thus by Stone duality, measure preserving homeomorphisms of $2^{\mathbb{N}}$. And by (4), $g^n = f$, while by (5) and (6), $f, g \in \bigcap_i V_i = C$. Thus, f and g belong to the comeagre conjugacy class and are therefore mutually conjugate. \square

Proposition 5. *Let G be a Polish group with a comeagre conjugacy class. Then the generic element of G is conjugate to its inverse.*

Proof. Let C be the comeagre conjugacy class of G . Then also C^{-1} is comeagre, so must intersect C in some point g . Thus both g and g^{-1} are generic and hence conjugate. Now, being conjugate with your inverse is a conjugacy invariant property and thus holds generically in G . \square

Theorem 6. *Let $n \neq 0$. Then the generic measure preserving homeomorphism of Cantor space is conjugate with its n 'th power and hence has roots of all orders.*

Thus, for the generic measure preserving homeomorphism g , there is an action of $(\mathbb{Q}, +)$ by measure preserving homeomorphisms of $2^{\mathbb{N}}$ such that g is the action by $1 \in \mathbb{Q}$.

Proof. We know that the generic g is conjugate to all its positive powers and to g^{-1} . But then g^{-1} is generic and thus conjugate to $(g^{-1})^n = g^{-n}$, whence g is conjugate with g^{-n} , $n \geq 1$.

So suppose g is generic and $n \geq 1$. Then, there is some f such that $(fgf^{-1})^n = fg^n f^{-1} = g$, and hence g has a generic n 'th root, namely, fgf^{-1} . This means that we can define a sequence $g = g_1, g_2, \dots$ of generic elements such that g_{n+1} is an $n+1$ 'th root of g_n , $g_{n+1}^{n+1} = g_n$. The following therefore defines an embedding of $(\mathbb{Q}, +)$ into $\text{Homeo}(2^{\mathbb{N}}, \mu)$ with $1 = \frac{1}{1!} \mapsto g_1$,

$$\frac{k}{n!} \mapsto g_n^k,$$

$k \in \mathbb{Z}$, $n \geq 1$. \square

3. POWERS OF GENERIC ISOMETRIES

3.1. Free amalgams of metric spaces. Suppose \mathbf{A} and $\mathbf{B}_1, \dots, \mathbf{B}_n$ are non-empty finite metric spaces and $\iota_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$ is an isometric embedding for each i . We define the *free amalgam* $\bigsqcup_{\mathbf{A}} \mathbf{B}_i$ of $\mathbf{B}_1, \dots, \mathbf{B}_n$ over \mathbf{A} and the embeddings ι_1, \dots, ι_n as follows.

Denote by d_i the metric on \mathbf{B}_i for each i and let $\mathbf{C}_i = \mathbf{B}_i \setminus \iota_i[\mathbf{A}]$. By renaming elements, we can suppose that $\mathbf{C}_1, \dots, \mathbf{C}_n$ and \mathbf{A} are pairwise disjoint.

We then let the universe of $\bigsqcup_{\mathbf{A}} \mathbf{B}_i$ be $\mathbf{A} \cup \bigcup_{i=1}^n \mathbf{C}_i$ and define the metric ∂ by the following conditions

- (1) $\partial(x, y) = d_i(\iota_i x, \iota_i y)$ for $x, y \in \mathbf{A}$,
- (2) $\partial(x, y) = d_i(\iota_i x, y)$ for $x \in \mathbf{A}$ and $y \in \mathbf{C}_i$,
- (3) $\partial(x, y) = d_i(x, y)$ for $x, y \in \mathbf{C}_i$,
- (4) $\partial(x, y) = \min_{z \in \mathbf{A}} d_i(x, \iota_i z) + d_j(\iota_j z, y)$ for $x \in \mathbf{C}_i$ and $y \in \mathbf{C}_j$, $i \neq j$.

We notice first that in (1) the definition is independent of i since each ι_i is an isometry. Also, a careful checking of the triangle inequality shows that this indeed defines a metric ∂ on $\mathbf{A} \cup \bigcup_{i=1}^n \mathbf{C}_i$.

We define for each i an isometric embedding $\pi_i: \mathbf{B}_i \hookrightarrow \bigsqcup_{\mathbf{A}} \mathbf{B}_l$ by

- $\pi_i(x) = x$ for $x \in \mathbf{C}_i$,
- $\pi_i(\iota_i x) = x$ for $x \in \mathbf{A}$.

Notice that in this way the following diagram commutes

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\iota_i} & \mathbf{B}_i \\ \iota_j \downarrow & & \downarrow \pi_i \\ \mathbf{B}_j & \xrightarrow{\pi_j} & \bigsqcup_{\mathbf{A}} \mathbf{B}_l \end{array}$$

3.2. Roots of isometries.

Proposition 7. *Let $\mathbf{A} \subseteq \mathbf{B}$ be finite rational metric spaces, f and isometry of \mathbf{A} and g an isometry of \mathbf{B} leaving \mathbf{A} invariant and such that $f^n = g|_{\mathbf{A}}$ for some $n \geq 1$. Then there is a finite rational metric space $\mathbf{D} \supseteq \mathbf{B}$ and an isometry h of \mathbf{D} such that h^n leaves \mathbf{B} invariant and $h^n|_{\mathbf{B}} = g$.*

Proof. Let $\mathbf{B}_1 = \dots = \mathbf{B}_n = \mathbf{B}$ and define isometric embeddings $\iota_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$ by

$$\iota_i(x) = f^{-i}(x).$$

To distinguish between the different copies of \mathbf{B} , we let for $x \in \mathbf{B} \setminus \mathbf{A}$, x^i denote the copy of x in $\mathbf{C}_i = \mathbf{B}_i \setminus \iota_i[\mathbf{A}] = \mathbf{B}_i \setminus \mathbf{A}$. Note also that $\mathbf{B} = \mathbf{B}_1 = \dots = \mathbf{B}_n$ all have the same metric, which we denote by d . We now define h on $\bigsqcup_{\mathbf{A}} \mathbf{B}_l$ as follows.

- $h(x) = f(x)$ for $x \in \mathbf{A}$,
- $h(x^i) = x^{i+1}$ for $x \in \mathbf{B} \setminus \mathbf{A}$ and $1 \leq i < n$,
- $h(x^n) = (gx)^1$ for $x \in \mathbf{B} \setminus \mathbf{A}$.

Now, obviously, h is a permutation of \mathbf{A} and for $1 \leq i < n$, h is a bijection between \mathbf{C}_i and \mathbf{C}_{i+1} . Moreover, h is a bijection between \mathbf{C}_n and \mathbf{C}_1 . Therefore, h is a permutation of $\bigsqcup_{\mathbf{A}} \mathbf{B}_l$. We check that h is 1-Lipschitz.

Suppose first that $x, y \in \mathbf{A}$. Then

$$\begin{aligned} \partial(hx, hy) &= \partial(fx, fy) \\ &= d(\iota_i fx, \iota_i fy) \\ &= d(f^{-i} fx, f^{-i} fy) \\ &= d(f^{1-i} x, f^{1-i} y) \\ &= d(f^{-i} x, f^{-i} y) \\ &= d(\iota_i x, \iota_i y) \\ &= \partial(x, y). \end{aligned}$$

Also, h is clearly an isometry between \mathbf{C}_i and \mathbf{C}_{i+1} for $1 \leq i < n$. So consider the case \mathbf{C}_n . Fix $x, y \in \mathbf{B} \setminus \mathbf{A}$. Then

$$\begin{aligned} \partial(h(x^n), h(y^n)) &= \partial((gx)^1, (gy)^1) \\ &= d(gx, gy) \\ &= d(x, y) \\ &= \partial(x^n, y^n). \end{aligned}$$

Now, if $x \in \mathbf{A}$, $y \in \mathbf{B} \setminus \mathbf{A}$, and $1 \leq i < n$, then

$$\begin{aligned} \partial(h(x), h(y^i)) &= \partial(fx, y^{i+1}) \\ &= d(\iota_{i+1} fx, y) \\ &= d(f^{-(i+1)} fx, y) \\ &= d(f^{-i} x, y) \\ &= d(\iota_i x, y) \\ &= \partial(x, y^i). \end{aligned}$$

Also, if $x \in \mathbf{A}$, $y \in \mathbf{B} \setminus \mathbf{A}$, then

$$\begin{aligned}
 \partial(h(x), h(y^n)) &= \partial(fx, (gy)^1) \\
 &= d(\iota_1 fx, gy) \\
 &= d(f^{-1}fx, gy) \\
 &= d(x, gy) \\
 &= d(g^{-1}x, y) \\
 &= d(f^{-n}x, y) \\
 &= d(\iota_n x, y) \\
 &= \partial(x, y^n).
 \end{aligned}$$

And finally, if $x, y \in \mathbf{B} \setminus \mathbf{A}$ and $1 \leq i < j \leq n$, we pick $z \in \mathbf{A}$ such that the distance $\partial(x^i, y^j)$ is witnessed by z , i.e.,

$$\partial(x^i, y^j) = d(x, \iota_i z) + d(\iota_j z, y) = d(x, f^{-i}z) + d(f^{-j}z, y).$$

Assume first that $j < n$. Then

$$\begin{aligned}
 \partial(h(x^i), h(y^j)) &= \partial(x^{i+1}, y^{j+1}) \\
 &\leq d(x, \iota_{i+1}fz) + d(\iota_{j+1}fz, y) \\
 &= d(x, f^{-i}z) + d(f^{-j}z, y) \\
 &= \partial(x^i, y^j).
 \end{aligned}$$

And if $j = n$, we have

$$\begin{aligned}
 \partial(h(x^i), h(y^n)) &= \partial(x^{i+1}, (gy)^1) \\
 &\leq d(x, \iota_{i+1}fz) + d(\iota_1 fz, gy) \\
 &= d(x, f^{-i}z) + d(z, f^n y) \\
 &= d(x, f^{-i}z) + d(f^{-n}z, y) \\
 &= \partial(x^i, y^n).
 \end{aligned}$$

Thus, h is an isometry of $\bigsqcup_{\mathbf{A}} \mathbf{B}_l$.

Now see g and f as isometries of the first copy \mathbf{B}_1 of \mathbf{B} , i.e., $g(x^1) = (gx)^1$ for $x^1 \in \mathbf{C}_1$. Let $\pi_1: \mathbf{B}_1 \hookrightarrow \bigsqcup_{\mathbf{A}} \mathbf{B}_l$ be the canonical isometric embedding defined by

- $\pi_1(x^1) = x^1$ for $x^1 \in \mathbf{C}_1$,
- $\pi_1(\iota_1 x) = x$ for $x \in \mathbf{A}$.

To finish the proof, we need to show that the following diagram commutes

$$\begin{array}{ccc}
 \mathbf{B}_1 & \xrightarrow{g} & \mathbf{B}_1 \\
 \pi_1 \downarrow & & \downarrow \pi_1 \\
 \bigsqcup_{\mathbf{A}} \mathbf{B}_l & \xrightarrow{h^n} & \bigsqcup_{\mathbf{A}} \mathbf{B}_l
 \end{array}$$

First, suppose $y = \iota_1 x \in \mathbf{A}$. Then

$$\begin{aligned}
 h^n \pi_1 y &= h^n \pi_1 \iota_1 x = h^n x = f^n x \\
 &= \pi_1 \iota_1 f^n x = \pi_1 f^{-1} f^n x = \pi_1 f^n f^{-1} x \\
 &= \pi_1 f^n \iota_1 x = \pi_1 f^n y = \pi_1 g y.
 \end{aligned}$$

Now suppose that $x \in \mathbf{B} \setminus \mathbf{A}$. Then

$$h^n \pi_1(x^1) = h^n(x^1) = h(x^n) = (gx)^1 = \pi_1(gx)^1 = \pi_1 g(x^1).$$

□

Proposition 8. *Let $n \geq 1$. Then the generic isometry of the rational Urysohn metric space is conjugate with its n 'th power.*

Proof. A basic open set in $\text{Iso}(\mathbb{Q}\mathbb{U})$ is on the form

$$U(h, \mathbf{A}) = \{g \in \text{Iso}(\mathbb{Q}\mathbb{U}) \mid g|_{\mathbf{A}} = h|_{\mathbf{A}}\},$$

where \mathbf{A} is a finite subspace of $\mathbb{Q}\mathbb{U}$ and $h \in \text{Iso}(\mathbb{Q}\mathbb{U})$. We claim that for any $U(h, \mathbf{A})$ there is some finite $\mathbf{B} \subseteq \mathbb{Q}\mathbb{U}$ containing \mathbf{A} and some isometry k leaving \mathbf{B} invariant, such that $U(k, \mathbf{B}) \subseteq U(h, \mathbf{A})$. For if h and \mathbf{A} are given, choose by Theorem 1 some finite $\mathbf{B} \subseteq \mathbb{Q}\mathbb{U}$ containing both \mathbf{A} and $h(\mathbf{A})$ such that the partial isometry $h: \mathbf{A} \rightarrow h(\mathbf{A})$ of $\mathbf{A} \cup h(\mathbf{A})$ extends to an isometry \hat{h} of \mathbf{B} . Let k be any isometry of $\mathbb{Q}\mathbb{U}$ that extends \hat{h} . Then \mathbf{B} is k -invariant while $U(k, \mathbf{B}) \subseteq U(h, \mathbf{A})$.

Again, if k is an isometry of some finite $\mathbf{B} \subseteq \mathbb{Q}\mathbb{U}$, we let $U(k, \mathbf{B}) = \{g \in \text{Iso}(\mathbb{Q}\mathbb{U}) \mid g|_{\mathbf{B}} = k\}$.

Let now C be the comeagre conjugacy class of $\text{Iso}(\mathbb{Q}\mathbb{U})$ and find dense open sets $V_i \subseteq \text{Iso}(\mathbb{Q}\mathbb{U})$ such that $C = \bigcap_i V_i$. Enumerate the points of $\mathbb{Q}\mathbb{U}$ as a_0, a_1, a_2, \dots . We shall define a sequence of finite subsets $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \mathbf{A}_2 \subseteq \dots \subseteq \mathbb{Q}\mathbb{U}$ and isometries g_i and f_i of \mathbf{A}_i such that

- (1) $a_i \in \mathbf{A}_{i+1}$,
- (2) g_{i+1} extends g_i ,
- (3) f_{i+1} extends f_i ,
- (4) $g_i^n = f_i$,
- (5) $U(g_{i+1}, \mathbf{A}_{i+1}) \subseteq V_i$,
- (6) $U(f_{i+1}, \mathbf{A}_{i+1}) \subseteq V_i$.

To begin, let $\mathbf{A}_0 = \emptyset$ with trivial isometries $g_0 = f_0$. So suppose \mathbf{A}_i , g_i , and f_i are defined. We let $\mathbf{B} \subseteq \mathbb{Q}\mathbb{U}$ be a finite subset containing both a_i and \mathbf{A}_i and such that there is some isometry h of \mathbf{B} extending g_i . As V_i is dense open we can find some $U(k, \mathbf{C}) \subseteq V_i$, where $\mathbf{C} \subseteq \mathbb{Q}\mathbb{U}$ is a k -invariant finite set containing \mathbf{B} and k extends h . Again, as V_i is dense open, we can find some $U(p, \mathbf{D}) \subseteq V_i$, where $\mathbf{D} \subseteq \mathbb{Q}\mathbb{U}$ is a finite set containing \mathbf{C} , p an isometry of $\mathbb{Q}\mathbb{U}$ leaving \mathbf{D} invariant and extending $k^n|_{\mathbf{C}}$.

Now, by Proposition 7, we can find a finite subset $\mathbf{E} \subseteq \mathbb{Q}\mathbb{U}$ containing \mathbf{D} and an isometry q of \mathbf{E} extending $k|_{\mathbf{C}}$ such that q^n extends $p|_{\mathbf{D}}$. Finally, set $\mathbf{A}_{i+1} = \mathbf{E}$,

$$g_{i+1} = q \supseteq k|_{\mathbf{C}} \supseteq h \supseteq g_i,$$

and

$$f_{i+1} = q^n \supseteq p|_{\mathbf{D}} \supseteq k^n|_{\mathbf{C}} \supseteq g_i^n = f_i.$$

Then $U(g_{i+1}, \mathbf{A}_{i+1}) \subseteq U(k, \mathbf{C}) \subseteq V_n$ and $U(f_{i+1}, \mathbf{A}_{i+1}) \subseteq U(p, \mathbf{D}) \subseteq V_n$.

Set now $g = \bigcup_i g_i$ and $f = \bigcup_i f_i$. By (1), (2), and (3), f and g are isometries of $\mathbb{Q}\mathbb{U}$. And by (4), $g^n = f$, while by (5) and (6), $f, g \in \bigcap_i V_i = C$. Thus, f and g belong to the comeagre conjugacy class and are therefore mutually conjugate. □

Now in exactly the same way as for measure preserving homeomorphisms, we can prove

Theorem 9. *Let $n \neq 0$. Then the generic isometry of the rational Urysohn metric space is conjugate with its n 'th power and hence has roots of all orders.*

Thus, for the generic isometry g , there is an action of $(\mathbb{Q}, +)$ by isometries of $\mathbb{Q}\mathbb{U}$ such that g is the action by $1 \in \mathbb{Q}$.

4. COMEAGRE CONJUGACY CLASSES AND BAIRE CATEGORY

We now set up the framework allowing us to pass between a Polish group and a dense subgroup with a comeagre conjugacy class.

Definition 10. *Let $\pi: X \rightarrow Y$ be a Borel map between Polish spaces. We say that π is categorical if the following conditions hold.*

- (1) *If $A \subseteq X$ is a nonmeagre analytic set, then $\pi(A) \subseteq Y$ is nonmeagre,*
- (2) *if $A \subseteq X$ is a comeagre analytic set, then $\pi(A) \subseteq Y$ is comeagre,*
- (3) *if $B \subseteq Y$ is a meagre analytic set, then $\pi^{-1}(B) \subseteq X$ is meagre,*
- (4) *if $B \subseteq Y$ is a nonmeagre analytic set, then $\pi^{-1}(B) \subseteq X$ is nonmeagre,*
- (5) *and if $B \subseteq Y$ is a comeagre analytic set, then $\pi^{-1}(B) \subseteq X$ is comeagre.*

We say that π is strongly categorical if, moreover,

- (6) *If $A \subseteq X$ is a meagre analytic set, then $\pi(A) \subseteq Y$ is meagre.*

We notice that (1) \Leftrightarrow (3) \Leftrightarrow (5) and (2) \Leftrightarrow (4). But (2) \nRightarrow (1) \nRightarrow (2). However, if $\pi(X)$ is comeagre in Y , then (6) \Rightarrow (2).

Lemma 11. *Let $\pi: X \rightarrow Y$ be a surjective, continuous, and open map between Polish spaces. Then π is categorical.*

Proof. (5) follows from exercise (8.45) in Kechris [7]. So to see (2), assume towards a contradiction that $A \subseteq X$ is comeagre, but $\pi(A)$ is not comeagre. Then, pick some non-empty open $U \subseteq Y$ in which $\pi(A)$ is meagre and let $X_0 = \pi^{-1}(U)$, which is non-empty and open. The map $\pi: X_0 \rightarrow U$ is then surjective, continuous, and open, whereby (5) holds for this map. But then $\pi(A \cap X_0)$ is nonmeagre in U being the image of the nonmeagre set $A \cap X_0$ and using the equivalence of (1) and (5). This contradicts that $\pi(A)$ is meagre in U . \square

Before we state the next result, recall that, by a theorem of Marker and Sami [2], if a Polish group G has a comeagre conjugacy class C , then C is G_δ in G and thus a Polish space in its induced topology. Moreover, G acts continuously and transitively on C by conjugation.

Proposition 12. *Suppose G is a Polish group with a comeagre conjugacy class C . Suppose also that an element of C is conjugate with all its non-zero powers. Then, for every $n \neq 0$, the map $\pi: C \rightarrow C$ defined by*

$$\pi(g) = g^n$$

is a surjective, continuous, and open G -map. In particular, π is categorical.

Proof. First notice that π is a continuous G -map, where G acts on C by conjugation,

$$\pi(ghg^{-1}) = (ghg^{-1})^n = gh^n g^{-1} = g\pi(h)g^{-1}.$$

Also, π is surjective. For if $f \in C$, we know that f and f^n are conjugate and thus for some $g \in G$, $gf^n g^{-1} = f$, whence $\pi(gfg^{-1}) = gf^n g^{-1} = f$. Finally, to see that π is open, suppose that $U \subseteq C$ is open and let $g \in U$. As G acts continuously on C by the conjugacy action, which we write as $h.g = hgh^{-1}$, we

can find some open neighbourhood V of 1 in G such that $V.g \subseteq U$. But then by Effros' Theorem (see Becker and Kechris [2]) the set $V.\pi(g)$ is open in C and $\pi(g) \in V.\pi(g) = \pi(V.g) \subseteq \pi(U)$. Thus $\pi(U)$ is open. \square

Proposition 13. *Suppose H is a Polish group with a dense subgroup G , which is Polish in a finer topology. Assume that G has a comeagre conjugacy class C whose elements are conjugate with their non-zero powers. Assume $n \neq 0$ and let $\pi: H \rightarrow H$ be the continuous H -map*

$$\pi(g) = g^n.$$

Then π is categorical.

Proof. Suppose towards a contradiction that $A \subseteq H$ is nonmeagre, while $\pi(A)$ is meagre. Find closed nowhere dense sets $F_n \subseteq H$ such that $\pi(A) \subseteq \bigcup_n F_n$. Then $\pi^{-1}(F_n)$ is closed and $A \subseteq \bigcup_n \pi^{-1}(F_n)$ is nonmeagre, whereby some $\pi^{-1}(F_n)$ must be nonmeagre and thus have non-empty interior U . Now, C is dense in H and thus $U \cap C$ is open in the topology of C , whereby $\pi(U \cap C)$ is nonmeagre and hence somewhere dense in C . Therefore, $\pi(U \cap C) \subseteq F_n$ is also somewhere dense in H , which is a contradiction. Thus, images of nonmeagre sets are nonmeagre.

We now need to show that also images of comeagre sets are comeagre. So suppose $A \subseteq H$ is comeagre and let $H_0 \leq H$ be a countable dense subgroup. Set

$$B = \bigcap_{g \in H_0} gAg^{-1},$$

which is comeagre and H_0 -invariant. But then $\pi(B)$ is nonmeagre and H_0 -invariant, whence, as the action of H_0 on H is topologically transitive, $\pi(B)$ must be comeagre. Thus also $\pi(A)$ is comeagre. \square

5. MEASURE PRESERVING AUTOMORPHISMS AND ISOMETRIES

We can now apply our Theorems 6 and 9 in combination with Proposition 13 to deduce results, the first of which is due to J.L.F. King, about respectively the automorphism group of a standard probability space and the isometry group of the Urysohn metric space.

Before stating the next result we recall the so called *weak topology* on the group $\text{Aut}([0, 1], \lambda)$ of Lebesgue measure preserving automorphisms of the unit interval: It is the weakest topology such that for all Borel sets $A, B \subseteq [0, 1]$ the map $g \mapsto \lambda(gA \triangle B)$ is continuous.

Theorem 14 (J.L.F. King [9]). *The generic measure preserving automorphism of the unit interval has roots of all orders.*

Proof. It is well-known that the group of Haar measure preserving homeomorphisms embeds continuously as a dense subgroup of the group H of measure preserving automorphisms of the unit interval (see, e.g., Halmos [5]). Thus, by Theorem 9 and Proposition 13, we know that for each $n \neq 0$, the map $\pi: H \rightarrow H$ given by $\pi(g) = g^n$ is categorical. In particular, $\pi(H)$ is comeagre in H and thus the generic element of H has an n 'th root. As $n \geq 1$ is arbitrary, this thus holds for all $n \geq 1$ simultaneously. \square

We should mention that there are now significantly better results known for the generic automorphism. Thus T. de la Rue and J. de Sam Lazaro [11] prove, in

response to a question of J.L.F. King [9], that the generic automorphism is the time 1 map of a measure preserving \mathbb{R} -flow and hence of course has roots of all orders.

Theorem 15. *The generic isometry of the Urysohn metric space has roots of all orders.*

Proof. We repeat the proof of Theorem 14 using that the isometry group of the rational Urysohn metric space embeds continuously and densely into the isometry group of the Urysohn metric space. \square

It is important to notice that there are in fact isometries of the Urysohn space without square roots. This is proved by J. Melleray in his thesis [10].

6. TOPOLOGICAL SIMILARITY AND ROHLIN'S LEMMA FOR ISOMETRIES

Suppose G is a Polish group and $f, g \in G$. We say that f and g are *topologically similar* if the topological groups $\langle f \rangle \leq G$ and $\langle g \rangle \leq G$ are isomorphic.

Notice first that any f is topologically similar to f^{-1} . For if $\psi(f^n) = f^{-n}$, then ψ is an involution homeomorphism, since inversion is continuous in G . Of course, if f and g have infinite order, then any isomorphic homeomorphism ϕ between $\langle f \rangle$ and $\langle g \rangle$ must send the generators to the generators and so either $\phi(f) = g$ or $\phi(f) = g^{-1}$. But then composing with ψ we can always suppose that $\phi(f) = g$.

Notice also that as each $\langle f \rangle$ is metrisable, f and g are topologically similar if and only if for all increasing sequences $(s_n) \subseteq \mathbb{N}$, $f^{s_n} \xrightarrow[n]{} 1$ if and only if $g^{s_n} \xrightarrow[n]{} 1$. Thus, in particular, topological similarity is a coanalytic equivalence relation. We notice also that topological similarity is really independent of the ambient group G . For example, if G is topologically embedded into another Polish group H , then f and g are topologically similar in G if and only if they are topologically similar in H .

Topological similarity is an obvious invariant for conjugacy, that is, if there is any way to make f and g conjugate in some Polish group, then they have to be topologically similar.

Of particular interest are the cases $G = \text{Aut}([0, 1], \lambda)$, $G = U(\ell_2)$, and $G = \text{Iso}(\mathbb{U})$. Here of course $\text{Aut}([0, 1], \lambda)$ sits inside of $U(\ell_2)$ via the Koopman representation and two measure preserving transformations f and g are said to be *spectrally equivalent* if they are conjugate in $U(\ell_2)$. By the spectral theorem, spectral equivalence is Borel. Also, topological similarity is coarser than spectral equivalence. To see this, we notice that mixing is not a topological similarity invariant, whereas it is a spectral invariant. For if f is mixing, then the automorphism $f \oplus \text{id}$ is a non-mixing transformation of $[0, 1] \oplus [0, 1]$ but generates a discrete subgroup of $\text{Aut}([0, 1] \oplus [0, 1], \lambda \oplus \lambda)$. So taking a transformation $h \in \text{Aut}([0, 1], \lambda)$ conjugate with $f \oplus \text{id}$, we see that f and h are topologically similar, since they both generate discrete groups. A survey of the closely related topic of topological torsion elements in topological groups is given by Dikranjan in [3].

Proposition 16. *Let G be a non-trivial Polish group such that for all infinite $S \subseteq \mathbb{N}$ the set $\mathbb{A}(S) = \{g \in G \mid \exists s \in S \ g^s = 1\}$ is dense. Then every topological similarity class of G is meagre.*

Moreover, for every infinite $S \subseteq \mathbb{N}$ the set

$$\mathbb{C}(S) = \{g \in G \mid \exists (s_n) \subseteq S \ g^{s_n} \xrightarrow[n]{} 1\}.$$

is dense G_δ and invariant under topological similarity.

Proof. Let $V_0 \supseteq V_1 \supseteq \dots$ be a basis of open neighbourhoods of the identity and set

$$\mathbb{B}(S, k) = \{g \in G \mid \exists n \in S \setminus [1, k] \ g^n \in V_k\},$$

which is open and dense since it contains $\mathbb{A}(S \setminus [1, k])$. Now set

$$\begin{aligned} \mathbb{C}(S) &= \{g \in G \mid \exists (s_n) \subseteq S \ g^{s_n} \xrightarrow{n} 1\} \\ &= \{g \in G \mid \forall k \exists n \in S \setminus [1, k] \ g^n \in V_k\} \\ &= \bigcap_k \mathbb{B}(S, k). \end{aligned}$$

Then $\mathbb{C}(S)$ is invariant under topological similarity and dense G_δ .

Now if some topological similarity class C was nonmeagre, then

$$C \subseteq \bigcap_{\substack{S \subseteq \mathbb{N} \\ \text{infinite}}} \mathbb{C}(S)$$

and hence for all $g \in C$, $g^n \xrightarrow{n} 1$, implying that $g = 1$, which is impossible. \square

Since by Rohlin's Lemma the sets $\{g \in \text{Aut}([0, 1], \lambda) \mid \exists s \in S \ g^s = 1\}$ are dense in $\text{Aut}([0, 1], \lambda)$ for all infinite $S \subseteq \mathbb{N}$, we have

Corollary 17. *Every topological similarity class is meagre in $\text{Aut}([0, 1], \lambda)$.*

This improves a result of Rohlin saying that all conjugacy classes are meagre in $\text{Aut}([0, 1], \lambda)$. We clearly see the importance of Rohlin's Lemma in these matters. However, interestingly, Rohlin's Lemma can also be used to prove the existence of dense conjugacy classes in $\text{Aut}([0, 1], \lambda)$.

We now have the following analogue of Rohlin's Lemma for isometries of the Urysohn metric space.

Proposition 18 (Rohlin's Lemma for isometries). *Suppose $S \subseteq \mathbb{N}$ is infinite. Then the set*

$$\{g \in \text{Iso}(\mathbb{U}) \mid \exists n \in S \ g^n = 1\}$$

is dense in $\text{Iso}(\mathbb{U})$.

A *finite circular order* is a finite subset \mathbb{F} of the unit circle S^1 . If $x \in \mathbb{F}$, we denote by x^+ the first $y \in \mathbb{F}$ encountered by moving counterclockwise around S^1 beginning at x . We then denote x by y^- , i.e., $x^+ = y$ if and only if $y^- = x$.

Lemma 19. *Suppose h is an isometry of \mathbb{U} and $\delta > 0$. Then for all finite $\mathbf{A} \subseteq \mathbb{U}$ there is an isometry f of \mathbb{U} such that $d(f(a), h(a)) \leq \delta$ for all $a \in \mathbf{A}$ while $d(a, f(b)) \geq \delta$ for all $a, b \in \mathbf{A}$.*

Proof. Let $\mathbf{B} = \mathbf{A} \cup h[\mathbf{A}]$ and let $\mathbf{C} = \mathbf{B} \times \{0, \delta\}$ be equipped with the ℓ_1 -metric $d_1((b, x), (b', y)) = d(b, b') + |x - y|$. Clearly, \mathbf{B} is isometric with $\mathbf{B} \times \{0\}$ and $\mathbf{B} \times \{\delta\}$, so we can assume that \mathbf{B} is actually $\mathbf{B} \times \{0\} \subseteq \mathbf{C} \subseteq \mathbb{U}$. Now, let f be any isometry of \mathbb{U} such that $f(a, 0) = (h(a), \delta)$ for $a \in \mathbf{A}$. \square

Now for the proof of Proposition 18.

Proof. Suppose $\mathbf{A} \subseteq \mathbb{U}$ is finite, h an isometry of \mathbb{U} , and $\epsilon > 0$. We wish to find some isometry g such that $d(g(a), h(a)) < \epsilon$ for all $a \in \mathbf{A}$ and such that for some $s \in S$, $g^s = 1$. Find first some f such that $d(f(a), h(a)) < \epsilon$ for all $a \in \mathbf{A}$ while $d(a, f(b)) > \epsilon/2$ for all $a, b \in \mathbf{A}$. It is therefore enough to find some g that agrees with f on \mathbf{A} while $g^s = 1$ for some $s \in S$.

We let $\Delta = \text{diam}(\mathbf{A} \cup f[\mathbf{A}])$ and $\delta = \min(d(x, f(y)) \mid x, y \in \mathbf{A})$. Fix a number $s \in S$ such that $\delta \cdot (s - 2) \geq \Delta$ and take a finite circular order \mathbb{F} of cardinality s . Now let

$$\mathbf{B} = \{a \bullet x \mid a \in \mathbf{A} \text{ \& } x \in \mathbb{F}\},$$

where $a \bullet x$ are formally new points.

A *path* in \mathbf{B} is a sequence $p = (a_0 \bullet x_0, a_1 \bullet x_1, \dots, a_n \bullet x_n)$ where $n \geq 1$ and such that for each i , x_{i+1} is either x_i^- , x_i , or x_i^+ . We define the *length* of a path by

$$\ell(p) = \sum_{i=0}^{n-1} \rho(a_i \bullet x_i, a_{i+1} \bullet x_{i+1}),$$

where

$$\rho(a \bullet x, b \bullet y) = \begin{cases} d(a, b), & \text{if } y = x; \\ d(a, f(b)), & \text{if } y = x^+; \\ d(f(a), b), & \text{if } y = x^-, \end{cases}$$

and put $|p| = n + 1$.

Therefore, if \check{p} denotes the reverse path of p and $p \cdot q$ the concatenation of two paths (whenever it is defined), then $\ell(\check{p}) = \ell(p)$ and $\ell(p \cdot q) = \ell(p) + \ell(q)$. Thus, ℓ is the distance function in a finite graph with weighted edges and hence the following defines a metric on \mathbf{B}

$$D(a \bullet x, b \bullet y) = \inf \{ \ell(p) \mid p \text{ is a path with initial point } a \bullet x \text{ and end point } b \bullet y \}.$$

We say that two paths are *equivalent* if they have the same initial point and the same end point. We also say that a path p is *positive* if either $p = (a \bullet x, b \bullet x)$ for some $x \in \mathbb{F}$ or $p = (a_0 \bullet x_0, a_1 \bullet x_1, \dots, a_n \bullet x_n)$, where $x_{i+1} = x_i^+$ for all i . Similarly, p is *negative* if either $p = (a \bullet x, b \bullet x)$ for some $x \in \mathbb{F}$ or $p = (a_0 \bullet x_0, a_1 \bullet x_1, \dots, a_n \bullet x_n)$, where $x_{i+1} = x_i^-$ for all i . So p is positive if and only if \check{p} is negative. Notice also that if p is positive, then $\ell(p) \geq \delta \cdot (|p| - 2)$.

Lemma 20. *For every path p there is an equivalent path q , with $\ell(q) \leq \ell(p)$, which is either positive or negative.*

Proof. If p is not either positive or negative, then there is a segment of p of one of the following forms

$$\begin{array}{ll} (1) & (a \bullet x, b \bullet x, c \bullet x), & (5) & (a \bullet x, b \bullet x^-, c \bullet x), \\ (2) & (a \bullet x^+, b \bullet x, c \bullet x), & (6) & (a \bullet x, b \bullet x, c \bullet x^+), \\ (3) & (a \bullet x^-, b \bullet x, c \bullet x), & (7) & (a \bullet x, b \bullet x, c \bullet x^-). \\ (4) & (a \bullet x, b \bullet x^+, c \bullet x), & \end{array}$$

We replace these by respectively

$$\begin{array}{ll} (1') & (a \bullet x, c \bullet x), & (5') & (a \bullet x, c \bullet x), \\ (2') & (a \bullet x^+, c \bullet x), & (6') & (a \bullet x, c \bullet x^+), \\ (3') & (a \bullet x^-, c \bullet x), & (7') & (a \bullet x, c \bullet x^-), \\ (4') & (a \bullet x, c \bullet x), & \end{array}$$

and see that by the triangle inequality for d we can only decrease the value of ℓ . For example, in case (3), we see that

$$\begin{aligned} \rho(a \bullet x^-, b \bullet x) + \rho(b \bullet x, c \bullet x) &= d(a, f(b)) + d(b, c) \\ &= d(a, f(b)) + d(f(b), f(c)) \\ &\geq d(a, f(c)) \\ &= \rho(a \bullet x^-, c \bullet x). \end{aligned}$$

We can then finish the proof by induction on $|p|$. \square

We now claim that $D(a \bullet x, b \bullet x) = d(a, b)$. To see this, notice first that $D(a \bullet x, b \bullet x) \leq d(a, b)$. For the other inequality, let p be an either positive or negative path from $a \bullet x$ to $b \bullet x$. By symmetry, we can suppose p is positive. But then, unless $p = (a \bullet x, b \bullet x)$, we must have $|p| \geq s + 1$, whence also $\ell(p) \geq \delta \cdot (|p| - 2) \geq \delta \cdot (s - 1) \geq \Delta \geq d(a, b)$. A similar argument shows that $D(a \bullet x, b \bullet x^+) = d(a, f(b))$.

This shows that for any $x_0 \in \mathbb{F}$, $\mathbf{A} \cup f[\mathbf{A}]$ is isometric with $\mathbf{A} \times \{x_0, x_0^+\}$ by the function $a \mapsto a \bullet x_0$ and $f(a) \mapsto a \bullet x_0^+$. So we can just identify $\mathbf{A} \cup f[\mathbf{A}]$ with $\mathbf{A} \times \{x_0, x_0^+\}$. Notice also that the following mapping g is an isometry of \mathbf{B} :

$$a \bullet x \mapsto a \bullet x^+.$$

Moreover, it agrees with f on their common domain $\mathbf{A} \times \{x_0\}$. Realising \mathbf{B} as a subset of \mathbb{U} containing \mathbf{A} , we see that g acts by isometries on \mathbf{B} with $g^s = 1$. It then follows that g extends to a full isometry of \mathbb{U} still satisfying $g^s = 1$. \square

Corollary 21. *Every topological similarity class is meagre in $\text{Iso}(\mathbb{U})$.*

Again this strengthens a result of Kechris [4] saying that all conjugacy classes are meagre.

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Address of C. Rosendal:

Department of mathematics,
University of Illinois at Urbana-Champaign,
273 Altgeld Hall, MC 382,
1409 W. Green Street,
Urbana, IL 61801,
USA.
`rosendal@math.uiuc.edu`